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Thermal Property of a Two-Dimensional Partially Conducting Grate

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A grate consisting of equally spaced rectangular cylinders is embedded in a material of different property. The heat conduction across the grate is solved exactly by eigenfunction expansion and matching. The behavior of the eigenvalues and the corner singularities are analyzed. Effective conductivities are determined for various geometries and conductivity ratios. The results may be useful in the design of composites with a layer of spaced fiber bundles.

Nomenclature

A_0, A_n = constant coefficients
a = ratio of grate width to period
B_0, B_n = constant coefficients
b = ratio of gap width to period
C_1, C_2, C_3, C_4 = constants
d_1, d_2 = constants
E = functions, Eq. (26)
e_1, e_2 = constants
$f_1 f_2$ = constants
K_{1mn}, K_{2mn} = integrals, Eqs. (16) and (17)
L = half period
m, n = integers
p = constant, Eq. (45)
q = heat flux at infinity
r = polar coordinate
T = normalized temperature
x, y = Cartesian coordinates
α_n = $n\pi$
β_n = eigenvalue from Eq. (9)
γ = constant power
θ = polar coordinate
$\bar{\kappa}$ = effective conductivity of grate
κ_1 = conductivity of grate material
κ_2 = conductivity of embedding material
λ = κ_1/κ_2
μ = constant, Eq. (11)
ν = constant, Eq. (24)
σ = $\bar{\kappa}/\kappa_2$

Introduction

CONSIDER a grate, consisting of a layer of aligned evenly spaced cylinders, embedded in a medium of different property. The grate could be a model of a capacitance mesh or it may represent reinforcing fibers in composite materials. It is a goal of this study to find the effective thermal conductivity normal to the grate.

Because of the geometry, the solution is very difficult to obtain. A crude homogenization method regards the grate as a homogeneous layer with an effective property obtained from the law of mixtures. There is sparse literature that solves the governing equation with the proper boundary conditions. Previous related works mainly involve doubly periodic arrays of cylinders, where a finite domain is repeated. For embedded circular cylinders, Keller and Sachs[1] used finite differences, whereas Perrins et al.[2] and Han and Cosner[3] matched cylindrical harmonics. Fogelholm and Grimvall[4] integrated numerically embedded square cylinders. Milton et al.,[5] Bao et al.,[6] and Wang[7] used eigenfunction expansions for arrays of square cylinders. The heat transfer across a layer of alternating dissimilar materials was investigated by Wang,[8] who also determined the thermal resistance of a partially conducting thin screen using the eigenfunction method.[9]

The present paper studies the thermal properties of the partially conducting grate whose cross section is shown in Fig. 1a. For this problem previous methods using eigenfunction expansions need to be modified. Although our Nomenclature is in terms of thermal conductivity, the results would apply to electrical conductivity and mass diffusivity as well.

Formulation

We normalize all lengths by L and the temperature by qL/κ_2. Because of symmetry we only need to consider the rectangular regions I and II enclosed by the dashed lines in Fig. 1b. Each region has its own Cartesian coordinates (x, y).

The governing equation is the Laplace equation with conditions requiring that the temperature and heat flux be continuous across the boundaries. Thus, for region I, $T_I(x, y)$ is symmetric with respect to y and, because of symmetry

$$\frac{\partial T_I}{\partial y}(x, 1) = 0 \tag{1}$$

The heat flux at infinity q yields

$$\frac{\partial T_I}{\partial x}(\infty, y) = 1 \tag{2}$$

The general solution satisfying these conditions are

$$T_I(x, y) = A_0 + x + \sum_{n=1}^{\infty} A_n \cos \alpha_n y e^{-\alpha_n x} \tag{3}$$

where $\alpha_n = n\pi$. Region II is more complicated with a jump in the derivative of temperature at the phase boundary. $T_{II}(x, y)$ is odd in x, even in y, and

$$\frac{\partial T_{II}}{\partial y}(x, 1) = 0 \tag{4}$$

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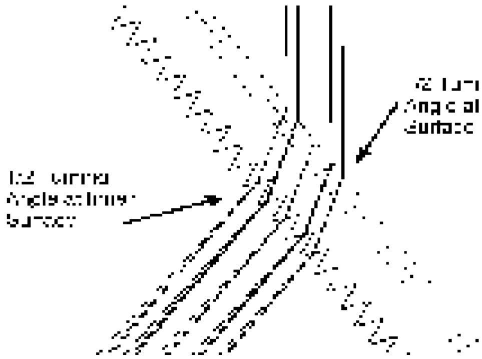

Fig. 1 Cross section of two-dimensional grate in a medium.

The general solution is

$$T_{\mathrm{II}}(x, y) = B_0 x + \sum_{n=1}^{\infty} B_n[e^{\beta_n(x-a)} - e^{-b_n(x+a)}]$$

$$\begin{cases} C_1 \cos[\beta_n(|y| - 1)] & b < |y| \le 1 \\ C_2 \cos(\beta_n y) & 0 \le |y| < b \end{cases} \tag{5}$$

Continuity of T_{II} at $y = \pm b$ gives

$$C_1 \cos[\beta_n(b - 1)] = C_2 \cos(\beta_n b) \tag{6}$$

Continuity of flux at $y = \pm b$ gives

$$\kappa_1 \frac{\partial T_{\mathrm{II}}}{\partial y}(x, b^+) = \kappa_2 \frac{\partial T_{\mathrm{II}}}{\partial y}(x, b^-) \tag{7}$$

or

$$\lambda C_1 \sin[\beta_n(b - 1)] = C_2 \sin(\beta_n b) \tag{8}$$

where $\lambda \equiv \kappa_1/\kappa_2$. From Eqs. (6) and (8) for nontrivial C_1 and C_2 one obtains the characteristic equation for β_n

$$\cos[\beta_n(b - 1)]\sin(\beta_n b) = \lambda \sin[\beta_n(b - 1)]\cos(\beta_n b) \tag{9}$$

The solution of Eq. (9) will be discussed later in the text. If $\cos(\beta_n b) = 0$ and $\cos[\beta_n(b - 1)] = 0$, from Eq. (8) one can set $C_1 = 1$, $C_2 = \lambda \sin[\beta_n(b - 1)]/\sin(\beta_n b)$. If $\cos(\beta_n b) \ne 0$ from Eq. (6), set $C_1 = 1$, $C_2 = \cos[\beta_n(b - 1)]/\cos(\beta_n b)$. Thus

$$T_{\mathrm{II}}(x, y) = B_0 x + \sum_{n=1}^{\infty} B_n[e^{\beta_n(x-a)} - e^{-\beta_n(x+a)}]$$

$$\begin{cases} \cos[\beta_n(|y| - 1)] & b < |y| \le 1 \\ \mu \cos(\beta_n y) & 0 \le |y| < b \end{cases} \tag{10}$$

where

$$\mu = \begin{cases} \lambda \dfrac{\sin[\beta_n(b - 1)]}{\sin(\beta_n b)} & \text{if } \cos(\beta_n b) = 0 \\ \dfrac{\cos[\beta_n(b - 1)]}{\cos(\beta_n b)} & \text{if } \cos(\beta_n b) \ne 0 \end{cases} \tag{11}$$

Lastly, regions I and II are to be matched along the boundary

$$T_{\mathrm{I}}(0, y) = T_{\mathrm{II}}(a, y) \tag{12}$$

$$\frac{\partial T_{\mathrm{I}}}{\partial x}(0, y) = \begin{cases} \lambda \dfrac{\partial T_{\mathrm{II}}}{\partial x}(a, y) & b < |y| \le 1 \\ \dfrac{\partial T_{\mathrm{II}}}{\partial x}(a, y) & 0 \le |y| < b \end{cases} \tag{13}$$

Equations (12) and (13) determine the coefficients A_0, A_m, B_0, and B_n

Solution

Equations (12) and (13) are to be solved by Fourier inversion. Integrating Eq. (12) from 0 to 1 gives

$$A_0 = B_0 a + \sum_n B_n(1 - e^{-2\beta_n a})\{\mu \sin(\beta_n b) - \sin[\beta_n(b - 1)]\}/\beta_n \tag{14}$$

Multiplying Eq. (12) by $\cos(\alpha_m y)$ and then integrating gives

$$\frac{A_m}{2} = \sum_n B_n(1 - e^{-2\beta_n a})[\mu K_{1mn} + K_{2mn}] \tag{15}$$

where

$$K_{1mn} = \int_0^b \cos(\beta_n y)\cos(\alpha_m y)\,dy$$

$$= \frac{\sin[(\alpha_m + \beta_n)b]}{2(\alpha_m + \beta_n)} + \begin{cases} \dfrac{\sin[(\alpha_m - \beta_n)b]}{2(\alpha_n - \beta_n)} & \alpha_m \ne \beta_n \\ \dfrac{b}{2} & \alpha_m = \beta_n \end{cases} \tag{16}$$

$$K_{2mn} = \int_b^1 \cos[\beta_n(1 - y)]\cos(\alpha_m y)\,dy$$

$$= \int_0^{1-b} \cos(\beta_n y)\cos[\alpha_m(1 - y)]\,dy = (-1)^m K_{1mn}|_{b\to 1-b} \tag{17}$$

Similar integrations on Eq. (13) yield

$$1 = B_0[b + \lambda(1 - b)] + \sum_n B_n(1 + e^{-2\beta_n a})\{\mu \sin(\beta_n b) - \lambda \sin[\beta_n(b - 1)]\} \tag{18}$$

$$-\frac{\alpha_m A_m}{2} = B_0 \frac{\sin(\alpha_m b)}{\alpha_m}(1 - \lambda) + \sum_n B_n \beta_n(1 + e^{-2\beta_n a})(\mu K_{1mn} + \lambda K_{2mn}) \tag{19}$$

Eliminate A_m from Eqs. (15) and (19)

$$0 = B_0 \frac{\sin(\alpha_m b)}{\alpha_m}(1 - \lambda) + \sum_n B_n\{[\beta_n + \alpha_m + (\beta_n - \alpha_m)e^{-2\beta_n a}]\mu K_{1mn} + [\beta_n\lambda + \alpha_m + (\beta_n\lambda - \alpha_m)e^{-\beta_n a}]K_{2mn}\} \tag{20}$$

We truncate the series to N terms. For n, $m = 1$ to N, Eqs. (18) and (20) give $N + 1$ linear equations for the $N + 1$ unknowns B_0 and B_n. Then A_0 and A_m are obtained from Eqs. (14) and (15) and the temperature distribution is completely determined. The accuracy can be ascertained by increasing N. Convergence is fairly fast. Normally $N = 10$ is sufficient for an error of 1% in temperature.

Eigenvalue β_n

For given λ and b, $\beta_n > 0$ can be found numerically from Eq. (9). However, for certain values of b an analytic solution is possible.

If $b = \frac{1}{2}$, Eq. (9) gives

$$(\lambda + 1)\cos(\beta_n/2)\sin(\beta_n/2) = 0 \tag{21}$$

Therefore, $\beta_n = n\pi$. Notice $\cos(\beta_n b) = 0$ for n odd; thus, both forms of Eq. (11) are needed. If $b = \frac{1}{4}$, and Eq. (9) becomes

$$\cos(\beta_n/4)\sin(\beta_n/4)[4\cos^2(\beta_n/4) - 3 + \lambda\langle 3 - 4\sin^2(\beta_n/4)\rangle] = 0 \tag{22}$$

If $\cos(\beta_n/4) = 0$ then $\beta_n = 4(n - \frac{1}{2})\pi$, if $\sin(\beta_n/4) = 0$, $\beta_n = 4n\pi$, if the bracket = 0, we find

$$\beta_n = 4(n - 1)\pi \pm 4\nu \tag{23}$$

$$\nu = \sin^{-1}\sqrt{\frac{1 + 3\lambda}{4(1 + \lambda)}} \tag{24}$$

Now we need to order the solutions of β_n according to their magnitudes. The result is

$$\beta_n = E[n - 4\,\mathrm{Integer}(n/4 - 0.1),\ 1 + \mathrm{Integer}(n/4 - 0.1)] \tag{25}$$

where

$$E[1, m] = 4(m - 1)\pi + 4\nu, \qquad E[2, m] = 4(m - \tfrac{1}{2})\pi$$
$$E[3, m] = 4m\pi - 4\nu, \qquad E[4, m] = 4m\pi \tag{26}$$

For $b = \frac{3}{4}$, Eq. (9) shows β_n has the same form, except all λ are replaced by $1/\lambda$. Notice one-quarter of the eigenvalues satisfy $\cos(\beta_n b) = 0$.

Similarly, analytic solutions for β_n can be found for $b = \frac{1}{3}$ and $\frac{2}{3}$, and for b = any rational number.

For very small λ, a perturbation solution can be found for any b, Eq. (9) shows either $\cos[\beta_n(b - 1)] \approx 0$ or $\sin(\beta_n b) \approx 0$. For the former we expand

$$\beta_n = \frac{(n - \frac{1}{2})}{b - 1}\pi + d_1\lambda + d_2\lambda^2 + \cdots \tag{27}$$

and substitute into Eq. (9). After some work we find

$$d_1 = \frac{1}{1 - b}\cot\left[\frac{b(n - \frac{1}{2})\pi}{b - 1}\right] \tag{28}$$

$$d_2 = \frac{-b}{(1 - b)^2}\cot\left[\frac{b(n - \frac{1}{2})\pi}{b - 1}\right]\csc^2\left[\frac{b(n - \frac{1}{2})\pi}{b - 1}\right] \tag{29}$$

The other set of solutions is

$$\beta_n = (n\pi/b) + e_1\lambda + e_2\lambda^2 + \cdots \tag{30}$$

$$e_1 = \frac{1}{b}\tan\left[\left(1 - \frac{1}{b}\right)n\pi\right] \tag{31}$$

$$e_2 = \frac{(b - 1)}{b^2}\tan\left[\left(1 - \frac{1}{b}\right)n\pi\right]\sec^2\left[\left(1 - \frac{1}{b}\right)n\pi\right] \tag{32}$$

For very large λ, the preceding expansion still holds with $\lambda \to 1/\lambda$, $b \to 1 - b$. For λ close to 1, we set $\lambda = 1 + \varepsilon$, and obtain

$$\beta_n = n\pi + f_1\varepsilon + f_2\varepsilon^2 + \cdots \tag{33}$$

where

$$f_1 = \sin(2bn\pi)/2 \tag{34}$$

$$f_2 = -\tfrac{1}{4}\sin(2bn\pi)[1 + (1 - 2b)\cos(2bn\pi)] \tag{35}$$

Isotherms

After the eigenvalues β_n are determined, the coefficients A_n and B_n are found from Eqs. (18) and (20). Typical isotherms are shown in Fig. 2 for $a = b = 0.5$ and various λ values. Only the first quadrants of regions I and II are shown. For $\lambda = 0$ (cylinders adiabatic) the temperature gradient is large in the gaps between the cylinders, where the net flux must go through. For $\lambda = 1$ (same conductivity) the isotherms are straight equally spaced vertical lines. For $\lambda \to \infty$ (cylinders perfectly conducting) the cylinders would have a uniform (zero) temperature. Figure 2c (for $\lambda = 200$) is a close approximation. Between these extremes the cylinders are partially conducting and a jump in temperature gradient on the boundary is observed.

Corner Singularity

The isotherms show no temperature extremes, but there might be very high conduction rates near the cylinder corners. We set polar coordinates (r, θ) at a corner and let the conductivity be κ_1 for $0 < \theta < (\pi/2)$ and κ_2 for $-(3\pi/2) < \theta < 0$. In a neighborhood close to the corner, let the temperatures be

$$T_1(r, \theta) = r^\gamma[c_1\cos(\gamma\theta) + c_2\sin(\gamma\theta)], \quad 0 < \theta < \pi/2$$
$$T_2(r, \theta) = r^\gamma[c_3\cos(\gamma\theta) + c_4\sin(\gamma\theta)], \quad -(3\pi/2) < \theta < 0 \tag{36}$$

Here r^γ is the leading power as $r \to 0$. Continuity in temperature gives

$$T_1(r, 0) = T_2(r, 0), \qquad T_1[r, (\pi/2)] = T_2[r, -(3\pi/2)] \tag{37}$$

Thus

$$c_1 = c_3 \tag{38}$$

$$c_1\cos(\pi\gamma/2) + c_2\sin(\pi\gamma/2) = c_3\cos(3\pi\gamma/2) - c_4\sin(3\pi\gamma/2) \tag{39}$$

Continuity of flux yields

$$\lambda\frac{\partial T_1}{\partial\theta}(r, 0) = \frac{\partial T_2}{\partial\theta}(r, 0), \qquad \lambda\frac{\partial T_1}{\partial\theta}\left(r, \frac{\pi}{2}\right) = \frac{\partial T_2}{\partial\theta}\left(r, -\frac{3\pi}{2}\right) \tag{40}$$

or

$$\lambda c_2 = c_4 \tag{41}$$

$$\lambda[-c_1\sin(\pi\gamma/2) + c_2\cos(\pi\gamma/2)] = c_3\sin(3\pi\gamma/2) + c_4\cos(3\pi\gamma/2) \tag{42}$$

For nontrivial solutions Eqs. (38), (39), (41), (42) give the characteristic equation

$$2\lambda[1 - \cos(\pi\gamma/2)\cos(3\pi\gamma/2)] + (\lambda^2 + 1)\sin(\pi\gamma/2)\sin(3\pi\gamma/2) = 0 \tag{43}$$

The solutions are

$$\gamma = 2n, \qquad \gamma = 2n \pm (2/\pi)\sin^{-1}(p), \quad n \text{ integer} \tag{44}$$

where

$$p = \sqrt{\frac{(3\lambda + 1)(\lambda + 3)}{4(\lambda + 1)^2}} \tag{45}$$

Fig. 2 Isotherms ($a = b = 0.5$). λ = a) 0, b) 0.2, c) 5, and d) 200.

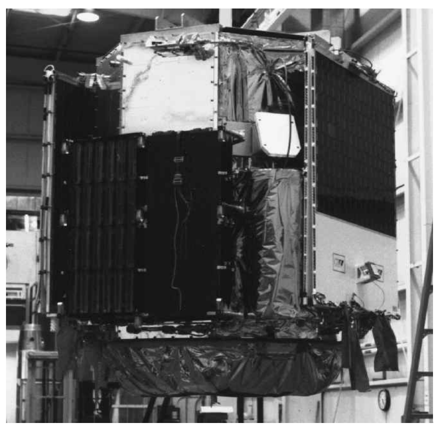

Fig. 3 Effective conductivity ratio for $\lambda = 0$. Solid lines are from the present work [Eq. (47)]. Dashed lines are from Ref. 9 or Eq. (50).

For T bounded as $r \to 0$, $\alpha \geq 0$. Because $\sqrt{3}/2 \leq p \leq 1$, the leading terms are

$$T \sim 1,\ r^{(2/\pi)\sin^{-1}(p)} \tag{46}$$

The heat flux is proportional to the derivative of Eq. (46) or to the radius to the power of $-1 + (2/\pi)\sin^{-1}(p)$, or, depending on λ, r^0 to $r^{-1/3}$, which is a very weak singularity.

Effective Conductivity

As x approaches infinity, the temperature T_I approaches $A_0 + x$. This means there is a temperature drop of $2A_0$ across the grate. Balancing heat flux we obtain

$$\sigma \equiv \bar{\kappa}/\kappa_2 = a/A_0 \tag{47}$$

where A_0, a function of λ, a, b, is from Eq. (14).

When a is very large, the grate is like closely spaced, long, stacked strips. We expect end effects are small in comparison with the total resistance. Thus, the rule of mixtures is a good approximation

$$\sigma = \lambda(1 - b) + b \tag{48}$$

When a is very small the grate is like a thin screen where the vertical strips are in the same plane. The only source for zero-thickness screens is from Wang.[9] Most of the results of that paper are not applicable here because the assumption is $a \to 0$, $\lambda \to \infty$ while $a\lambda$ remains a constant. However, an exact solution for $\lambda = 0$ was obtained by complex transform. The result is

$$\sigma = \frac{2a}{(4/\pi)\ell n[1/\sin(b\pi/2)]} \tag{49}$$

or, as a approaches zero with σ

$$\frac{\sigma}{a} = \frac{\pi}{2\ \ell n[1/\sin(b\pi/2)]} \tag{50}$$

Fig. 4 Effective conductivity ratio vs material conductivity ratio λ. b = a) 0.25, b) 0.5, and c) 0.75. The straight line for $a = \infty$ is from the rule of mixtures [Eq. (48)].

Figure 3 shows the results from the present work approach those of Ref. 9 as a approaches zero. Figure 4 shows the computed effective conductivity ratio vs λ for $b = 0.25, 0.5, 0.75$, and various a values. In general, σ increases with λ, but the increase is nonlinear, particularly for small a. We see that the formula for the law of mixtures Eq. (48) is valid for large a and λ close to unity. The effect of increasing b (or larger gaps), keeping other factors fixed, causes σ to be closer to 1. For other values of a, b, and λ one should be able to compute σ by the method described in this paper.

Discussion

The method of eigenfunction expansions and matching is highly efficient and well suited for this problem. One can also use finite differences, but the number of computations required would be more than the square of those used here. In addition, the infinite domain would be a hindrance to purely numerical means.

Bao et al.,[6] in their study of partially conducting square cylinders, obtained an eigenfunction expression similar to Eq. (10). Their work, however, is incomplete because only one form of Eq. (11) is retained. Thus, their convergence is questionable when $\cos(\beta_n b) = 0$ or when b is (or close to) rational numbers such as $\frac{1}{2}$, $\frac{3}{4}$, etc. The present paper gives the correct representations for the temperature field.

The corner singularity in the local heat flux is found to be weak. Our computed results show the singularity has little effect on the temperature or the effective conductivity.

The formula for effective conductivity using the law of mixtures fails for small a, and λ not close to unity. In these cases the effective conductivity should be computed as in the present paper. Figure 4 should be highly useful in the thermal design of partially conducting grates.

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